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A HIGH POWER LOSS MECHANISM IN SUPERCONDUCTING MICROWAVE CAVITIES

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The energy transmitted to a tenuous electron gas and subsequently delivered to the walls of a superconducting microwave TM_{010} mode cavity is computed. It is found that the cavity Q via this mechanism is proportional to various positive powers of the frequency, depending on the approximations used to solve the equations of motion. With an electron density of 10^1–10^2 cm^{-3} the Q at high power levels ($E \approx 10^7$ V/m) is approximately 10^8–10^9 at L-band frequencies. Cavities with radial nodes in the electric field strength are suggested as a means to reduce these losses.

1. INTRODUCTION

The use of superconducting microwave cavities in linear electron accelerators appears to be the next milestone in their development.[1] Such devices, because of their extremely low losses, can be continuously excited, thereby increasing the duty cycle to very close to unity. Additional reasons for a superconducting accelerator, which are related to the unity duty cycle, are increased energy resolution and accelerated electron current. Accordingly, experimental work has been directed toward fabrication of high Q superconducting cavities. In particular Turneaure and Viet[2] describe fabrication and processing techniques for superconducting TM_{010} mode niobium cavities at 8.6 GHz with high measured fields and unloaded Q's. Unloaded Q's as high as 10^{11} were measured at low field levels, and surface fields as high as 1080 Oe and 70 MV/m were measured with loaded Q's on the order of 10^{10}.

These results appeared to have demonstrated the usefulness of superconducting Nb for these devices. However in subsequent work it has been found that if Nb cavities were operated at the more practical frequency of 1.3 GHz the Q's at high field strengths were considerably less than those previously obtained. This result is so far unexplained.

In this communication we propose a loss mechanism which may contribute to the understanding of this observed behavior. We assume that a tenuous electron gas presumably produced by field emission is present in the cavity. As will be seen, the combined electric and magnetic fields in a TM_{010} mode cavity have such a configuration that it is possible to impart a net average radial motion to the electrons thereby driving them to the walls. The cavity Q based on this mechanism depends on positive powers of frequency and hence is more important at low frequencies.

Some effects of field emitted electrons in superconducting cavities have been discussed by others.[3–7] In particular Halbritter[3] discusses several mechanisms that can enhance field emission: Normal conducting regions such as protrusion spikes can be heated by large rf magnetic fields and thermionically assist more field-emitted electrons. Oxide layers and also dust particles can enhance field emission by reducing the effective work function. Usually field-emitted electrons from these mechanisms are copious enough to produce measurable frequency shift, x-ray emission, very poor Q's and even burnt spots in the cavities. As will be shown in the subsequent sections of this note electron densities of the order of 10 cm^{-3} or so can lead to Q's of the order of 10^9 at L-band frequencies. At these densities x-ray emission, burnt spots, frequency shifts are probably too small to be detected with any ease, however Turneaure[6] suggests his failure to obtain high Q's at 1.3 GHz may be due to field emitted electrons.

Further we point out that the mechanism proposed here is geometric in origin rather than a fundamental loss process as discussed by Rabinowitz[8] who considers resistive losses due to oscillating fluxoids trapped in the cavity surface.

2. MOTION OF THE ELECTRONS

We assume a TM_{010} mode cavity with radius a

and adopt cylindrical coordinates (r, θ, z). The fields for this mode are given by

$$E_z = E_0 J_0\left(\alpha_{01}\frac{r}{a}\right)\sin\omega t \qquad (1)$$

$$B_\theta = \frac{E_0}{\omega}\left(\frac{\alpha_{01}}{a}\right)J_1\left(\alpha_{01}\frac{r}{a}\right)\cos\omega t \qquad (2)$$

where the J's are ordinary Bessel functions, α_{01} is the first zero of $J_0(x) = 0$, and ω is the cavity resonant frequency. The equations of motion of a particle with charge q and mass m in these fields is (neglecting relativistic effects)

$$m\ddot{r} = mr\dot{\theta}^2 - q\dot{z}B_\theta \qquad (3)$$

$$m\ddot{z} = qE_z + q\dot{r}B_\theta \qquad (4)$$

$$mr^2\dot{\theta} = \text{const.} \qquad (5)$$

Since $mr^2\dot{\theta}$ is constant we take $\dot{\theta} = 0$ to simplify the analysis. Further we restrict ourselves to the region of the cavity near the axis where $\{\alpha_{01}(r/a)\} < 1$ and use the small argument formulas for the Bessel functions. As a final approximation we note that since the magnetic field is generated by the displacement current the magnetic force in (4) will be smaller than the electric force near the cavity axis. Thus we have

$$\dot{z} \approx -\frac{qE_0}{m\omega}\cos\omega t \qquad (6)$$

Substitution into (3) gives for the radial motion

$$\frac{d^2 r}{d\tau^2} - (h^2\cos^2\tau)r = 0 \qquad (7)$$

$\tau \equiv \omega t$ and $h \equiv qE_0/\sqrt{2}m\omega c$, with c the velocity of light. This is Mathieu's equation and since h^2 is a positive definite quantity the radial motion is unbounded.

At this point it is instructive to compute a typical value of h. With $\omega/2\pi = 10$ GHz, and $E_0 \approx 10^7$ V/m it is found that $h \approx 0.7\times10^{-1}$. The solutions to (7) are[9]

$$e^{is\tau}\sum_{n=-\infty}^{\infty} a_n e^{2in\tau} \quad \text{and} \quad e^{-is\tau}\sum_{n=-\infty}^{\infty} b_n e^{-2in\tau} \qquad (8)$$

When $h \ll 1$ we have $s^2 \approx -h^2/2$. Evaluating the coefficients it is found that $a_1 \approx (h^2/16)a_0$, $b_1 \approx (h^2/16)b_0$ so with the initial conditions $r(0) = r_0$ and $\dot{r}(0) = 0$ the radial velocity is given by

$$\dot{r} \approx \frac{1}{2\sqrt{2}} r_0 h\omega\left(\exp\frac{h\omega t}{\sqrt{2}} - \exp\frac{-h\omega t}{\sqrt{2}}\right) \qquad (9)$$

Calculation of the Q

An estimate of the cavity Q can now be obtained. We assume that the radial velocity acquired by an electron in the central region of the cavity is sufficient to carry it to the wall. It is also assumed that an equilibrium is established between electrons transported to the wall and free electrons produced in the cavity. The cavity Q is defined as the ratio of the peak energy stored in the cavity to the energy dissipated in one cycle. With the additional assumption that an electron only delivers the energy in its radial motion to the wall the Q is approximately

$$Q \approx \frac{\varepsilon_0 E_0{}^2}{n_e m(\dot{r})^2} \qquad (10)$$

where n_e is the mean electron density produced per cycle in the cavity. The radial velocity is evaluated at the end of one cycle. When $(2\pi/\sqrt{2})h < 1$ the exponential factors in (9) can be expanded to yield

$$\dot{r} \approx \pi r_0 \omega h^2 \qquad (11)$$

Defining $\eta \equiv r_0/a$ and since $\alpha_{01}/a = \omega/c$ for the TM_{010} mode (10) becomes

$$Q = \frac{4\varepsilon_0 m^3 c^2}{\pi^4 q^4 \eta^2 \alpha_{01}{}^2}\left(\frac{\omega^4}{n_e E_0{}^2}\right) \qquad (12)$$

If we take $\eta = 0.5$ the numerical factor is $\approx 1/4\pi^4 \times 10^{-7}$, thus $(\omega = 2\pi f)$

$$Q = 4\times10^{-7}\frac{f^4}{n_e E_0{}^2}. \qquad (13)$$

At a field strength of 10^7 V/m and $n = 10^2$ cm^{-3} the Q at 8.6 GHz is approximately 10^{11} (with $h \approx 0.08$). At the L-band frequency of 1.3 GHz, with the same density and field strength, $h \approx 0.5$ and the radial velocity can no longer be given by (11). Since $(2\pi/\sqrt{2})h \approx 2.2$ the radial velocity is given by

$$\dot{r} \approx \frac{1}{2\sqrt{2}} r_0 h\omega \exp\left(\frac{2\pi h}{\sqrt{2}}\right) \qquad (14)$$

and the Q by

$$Q = \frac{16}{\eta^2\alpha_{01}{}^2}\frac{m\varepsilon_0\omega^2}{n_e q^2}\exp\left(-\frac{2\pi q E_0}{m\omega c}\right). \qquad (15)$$

Evaluated at 1.3 GHz, Eq. (15) gives $Q \approx 2.3 \times 10^7$. However, since $h = 0.5$ for these parameters the approximations leading to (15) are in doubt. Equation (9) will still give the solution to (7) reasonably well since $a_1 = \frac{1}{64}a_0$ and $b_1 = \frac{1}{64}b_0$, however other factors must be considered. Defining $v_E = qE_0/m\omega$, the maximum axial velocity imparted from the electric field, we find $v_E/c = \sqrt{2}h \approx 0.7$ indicating that relativistic effects may be important. A more important error arises from the fact that when $h = 0.5$ the velocity obtained from (14) is about $2c$. But since $\dot{r}$ depends very strongly on h when $2\pi h/\sqrt{2} \approx 1$ it is found, for example, that with $h \approx 0.25$ the ratio $\dot{r}/c \approx 0.3$. Thus for the L-band parameters we conclude that (15) underestimates the Q by one or two orders of magnitude. Also the analysis relied on the magnetic force term in (4) being so small that it could be neglected when compared with the electric force. Evaluating $(\dot{r}B_\theta)\max/(E_z)\max$ with $\dot{r}$ given by (11) and taking $r \approx r_0$ it is easily shown that

$$(\dot{r}B_\theta)\max/(E_z)\max \approx \tfrac{1}{2}\pi\eta^2{\alpha_{01}}^2h^2 \tag{16}$$

which evaluates to 0.57 for the L-band parameters. This will also introduce error in (15). An estimate of the magnitude, however, cannot be made without more complete calculations.

It has been shown that those electrons in the central region of the cavity, near the axis, will acquire a non-zero average velocity in the radial direction and a Q was calculated assuming that the energy in the radial motion was delivered to the walls. Whether these electrons will continue to the cavity walls and deliver their energy is considered next.

The complete equations of motion including the spatial variations of the fields as given by the Bessel functions appear to be impossible to handle analytically without some approximation. We divide the cavity into three regions. A central region, near the axis, has been treated in the first section; an annular region, and a wall region. Near the wall the electric field becomes very small, therefore it is possible to neglect the electric force. Thus the equations of motion become

$$\ddot{z} = \lambda\dot{r}\cos\omega t \tag{17}$$

$$\ddot{r} = -\lambda\dot{z}\cos\omega t \tag{18}$$

with $\lambda = (qE_0/mc)J_1(\alpha_{01})$ since $r \approx a$. Integrating (17) by parts twice and making use of (18) we obtain

$$\dot{z}-\dot{Z} = \lambda\left\{\frac{\dot{r}}{\omega}\sin\omega t+\frac{\lambda}{\omega^2}\left[\frac{1}{2}\dot{z}\sin^2\omega t-\frac{1}{2}\int\ddot{z}\sin^2\omega t\,\mathrm{d}t\right]\right\} \tag{19}$$

It is obvious that λ/ω behaves as an expansion parameter if this process is continued indefinitely. Noting that when $\lambda/\omega \approx h \ll 1$ all terms but the first will be small thus

$$\dot{z} \approx \dot{Z}+\frac{\lambda}{\omega}\dot{r}\sin\omega t \tag{20}$$

where $\dot{Z}$ is the z component of velocity of an electron entering the wall region. Substitution into (18) gives an equation for the radial velocity, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r})+\frac{\lambda^2}{2\omega}\sin 2\omega t(\dot{r}) = -\lambda\dot{Z}\cos\omega t \tag{21}$$

which can be integrated to give

$$\dot{r} = C\exp\left[\frac{\lambda^2}{2\omega^2}\cos^2\omega t\right]-\frac{\lambda}{\omega}\dot{Z}\exp\left[-\frac{\lambda^2}{2\omega^2}\sin^2\omega t\right] \times\int\exp\left[\frac{\lambda^2}{2\omega^2}\sin^2\omega t\right]\cos\omega t\,\mathrm{d}(\omega t) \tag{22}$$

where C is a constant of integration. Again when $\lambda/\omega \ll 1$, (22) reduces to

$$\dot{r} \approx \dot{r}(0)\left[1-\frac{\lambda^2}{2\omega^2}\sin^2\omega t\right] \tag{23}$$

where $\dot{r}(0)$ is the radial velocity upon entering the wall region. Thus it is evident that if an electron passes through the annular region it will reach the wall.

In the annular region both the electric and magnetic forces as well as the explicit spatial variations must be retained thereby requiring use of a computer.

3. ADDITIONAL CONSIDERATIONS

The guiding center approximation can be used to compute particle orbits in magnetic fields that vary slowly in space and time. The oscillation center approximation[10] is a similar procedure which is applicable to the opposite limiting case, that of a very high frequency field. It is relatively easy to show that if the oscillation center, moving with velocity $\dot{r}$, does not carry the particle into appreciably different regions of the field during an oscillation period the motion of the particle is given by

$$\frac{d^2 r}{dt^2} = -\nabla\varphi \tag{24}$$

where

$$\varphi \equiv \frac{q^2}{m^2\omega^2}\left\langle\frac{E^2}{2}\right\rangle$$

and $\langle\,\rangle$ denotes an average over an oscillation period. It is immediately evident that the particle moves in the direction of decreasing electric field strengths. In a TM_{010} mode cavity electrons would thus move radially to the cavity walls confirming the hypothesis of the preceding sections.

Inserting the explicit electric field variation for the TM_{010} mode it is found that near the axis

$$\frac{d^2 r}{d\tau^2} = \tfrac{1}{2}h^2 r, \tag{25}$$

where r is now the position of the oscillation center. With initial conditions as before the velocity is

$$\dot{r} = \frac{r_0 h\omega}{2\sqrt{2}}\left(\exp\frac{h\omega t}{\sqrt{2}} - \exp\frac{-h\omega t}{\sqrt{2}}\right) \tag{26}$$

which agrees with our previous result (10).

The particle displacement after one period is given by

$$\delta r \equiv r - r_0 \approx 2\pi^2 h^2 r_0 \tag{27}$$

indicating that for small values of h it may take many cycles for an electron to reach the walls. However, as f is lowered h increases thereby allowing substantial displacements in one cycle.

It is also easy to show via the oscillation center approximation[10] that in the time average there is no energy exchange between the electromagnetic field and the particle (providing no wall gets in the way). There is simply an energy transformation between the oscillatory and translation kinetic energy, i.e. the axial oscillatory kinetic energy acquired near the cavity axis gradually gets transformed into radial kinetic energy. Thus if we assume that electrons in the cavity reach the walls because of the radial motion and again assume an equilibrium between free electrons produced in the cavity and those transported to the walls it is possible to calculate a Q. The average axial kinetic energy acquired near the axis of the cavity is $\frac{1}{4}(q^2 E_0{}^2/m\omega^2)$. If this energy gets transferred to the walls we find

$$Q \approx 2\frac{m\varepsilon_0}{n_e q^2}\omega^2 \tag{28}$$

Evaluated at L-band $Q \approx 4\times 10^8$ with a density of 10^2 cm^{-3}. This expression for the Q does not depend explicitly on E.

The most reasonable source of free electrons in the cavity seems to be field emission although secondary electron emission may also contribute. The field-emitted current is given by the Fowler–Nordheim equation, i.e.,

$$j = CE^2 \exp(-D/E)\ \mathrm{A/m^2}, \tag{29}$$

where

$$C = \frac{6.2\times 10^{-6}}{E_b}\left(\frac{E_m}{E_w}\right)^{1/2}\ \mathrm{A/V^2},$$

$$D = 6.8\times 10^9 E_w{}^{3/2}\ \mathrm{V/m},$$

and E_w is the work function, E_m the Fermi level and $E_b = E_w + E_m$. In formulas such as (29) the exponential term usually dominates and therefore will also dominate each of the formulas for the Q.

Halbritter[3] has estimated that prebreakdown field emitted electron currents are able to cause frequency shifts of about 1 kHz at 3 GHz. This corresponds to a mean electron density of 6×10^3 cm^{-3} and can be obtained from a contamination protrusion spike with an effective area of about $\sim 10^{-11}$ cm^2 at $E \approx 10^7$ V/m if the work function of the contamination is 2 eV.

This is considerably in excess of the densities discussed here. Note that a mean density of only *10* electrons per cm^3 will give a Q of 4×10^9 from (28). In view of the many assumptions (effective work function, field enhancement factor, etc.) that must be made to evaluate the field-emitted current from (29) no further speculations will be given here. Further, since field emission will occur only when E is close to its maximum value, the number of electrons released per cycle will be proportional to $j\omega^{-1}a^2$. If the cavity volume is taken to scale as a^3

the density of field-emitted electrons is independent of frequency.

It has been noted[11] that after tunneling through the surface potential barrier field-emitted electrons have very little velocity. In addition, since the electrons are emitted when E has its maximum values, it may be more appropriate to take

$$E_z(r, t) = E_z(r) \cos \omega t \tag{30}$$

with $\dot{z} = a$ at $t = 0$. This has been done. The resulting radial velocity after one period is unchanged, hence the Q is unaffected.

CONCLUSIONS

The mechanism proposed here appears to be a powerful loss mechanism in superconducting cavities and may explain the failure to obtain high Q's at high power levels in L-band cavities operating in the TM_{010} mode. The analysis is based on certain simplifying assumptions, therefore the expressions for the cavity Q should be interpreted as being suggestive of what to expect from this mechanism if the calculations were done exactly. The final results must be verified by integrating the exact equations of motion. Work on this aspect is under way. Further it is our supposition that this effect is mode dependent as the oscillation center approximation indicates, i.e., other modes with more complicated field patterns may not exhibit this loss or it may be greatly reduced; this is also being investigated.

As an experimental test of this hypothesis we suggest fabrication of a cavity oscillating in the TM_{020} mode. While preserving cylindrical symmetry this mode has a radial node in the electric field strength thereby creating a region that will trap electrons, prevent their motion to the cavity walls, and thus lead to a higher Q.

It should also be possible to detect the presence of field-emitted electrons by carefully examining the energy decay curve from which measurements of Q are obtained. Since the number of field-emitted electrons will decrease as the cavity field is lowered there should be departures from the exponential decay predicted by a Q independent of E.

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